Centre for Theoretical Physics
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TENTAMEN GENERAL RELATIVITY

tuesday, 23-01-2007, room 5111-0080, 9.00-12.00

Indicate at the first page clearly your name, address, date of birth, year of arrival and at every other page your name.

Question 1

Consider a Riemannian manifold with a covariantly constant metric $g_{ab}(x)$ and Christoffel symbols $\Gamma^c_{ab}(x) = \Gamma^c_{ba}(x)$.

(1.1) Give the definition of the covariant derivative $\nabla_a S$ of a scalar S, the covariant derivative $\nabla_a V_b$ of a covariant vector V_b and the covariant derivative $\nabla_a T^d_{bc}$ of a tensor T^d_{bc}.

(1.2) The covariant derivative of the metric g_{ab} is zero:

$$\nabla_c g_{ab}(x) = 0\,. \qquad (1)$$

Use this fact to derive an expression for the Christoffel symbol Γ^c_{ab} in terms of the metric and the derivative of the metric.

(1.3) Consider the metric of a 2–sphere S^2 with coordinates (r, ϕ):

$$ds^2 = \frac{4}{(1+r^2)^2}(dr^2 + r^2 d\phi^2)\,. \qquad (2)$$

One finds that the non-zero Christoffel symbols corresponding to the metric given in (2) are given by

$$\Gamma^r_{rr} = \frac{-2r}{1+r^2}\,, \qquad \Gamma^r_{\phi\phi} = \frac{-r(1-r^2)}{1+r^2}\,, \qquad \Gamma^\phi_{r\phi} = \frac{1-r^2}{r(1+r^2)}\,. \qquad (3)$$

Calculate the nonzero components of the Riemann tensor

$$R^c{}_{dab} = \partial_a \Gamma^c_{db} - \partial_b \Gamma^c_{da} + \Gamma^c_{ea}\Gamma^e_{db} - \Gamma^c_{eb}\Gamma^e_{da}\,. \tag{4}$$

(1.4) Show that the sphere S^2 is a maximally symmetric space, i.e.

$$R_{abcd} \propto (g_{ac}g_{bd} - g_{ad}g_{bc})\,. \tag{5}$$

Question 2

Consider the Schwarzschild metric (we take $c = 1$)

$$ds^2 = (1 - \frac{2m}{r})dt^2 - (1 - \frac{2m}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{6}$$

(2.1) Show that the Euler-Lagrange equations that follow from the Lagrangian

$$L = \frac{1}{2}\dot{x}^a\dot{x}^b g_{ab}(x) \tag{7}$$

are given by the geodesic equations

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0\,. \tag{8}$$

The dot $\dot{}$ indicates differentation with respect to the parameter s of the geodesic.

(2.2) Show that for constant r and $\theta = \pi/2$ the Schwarzschild metric (6) leads to the following geodesic equations

$$(1 - \frac{2m}{r})\dot{t} = k\,, \tag{9}$$

$$r^2\dot{\phi} = h\,, \tag{10}$$

$$\frac{m}{r^2}(\dot{t})^2 - r(\dot{\phi})^2 = 0\,, \tag{11}$$

with k and h constant.

(2.3) A light-ray follows a geodesic around a Schwarzschild black hole for constant $r = r_0$ and $\theta = \pi/2$. Determine the value of r_0.

Observer A finds himself in free fall in an orbit of constant radius $r_0 = 4m$. The orbit is in the plane $\theta = \pi/2$.

(2.4) Calculate the coordinate time Δt_A that observer A needs for one revolution.

Question 3

The Robertson-Walker metric for $k = 1, 0, -1$ can be written in the form (we take $c = 1$)

$$ds^2 = dt^2 - R^2(t)\{\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\}\,. \tag{12}$$

For the energy-momentum tensor of a perfect fluid the Einstein equations lead to the following relations between the function $R(t)$, the mass density $\rho(t)$ and the pressure $p(t)$:

$$\frac{(\dot{R})^2 + k}{R^2} = \tfrac{1}{3}\kappa\rho\,, \tag{13}$$

$$\dot{\rho} + 3(p+\rho)\frac{\dot{R}}{R} = 0\,. \tag{14}$$

The dot indicates a differentiation with respect to t and $\kappa = 8\pi G$ (G is Newton's constant).

Consider first the case of a flat universe with non-relativistic matter, i.e. $k = 0$ and $p = 0$.

(3.1) Show that ρR^3 is constant and that $\dot{R}^2 = A^2/R$ for some constant A. Determine this constant.

(3.2) Take as boundary condition that $R = 0$ at $t = 0$. Determine R as a function of t. Give the graph of the function $R(t)$.

We consider for the remaining part of this question the situation of a closed universe with ultra-relativistic matter, i.e. $k = 1$ and $p = \frac{1}{3}\rho$.

(3.3) Show that ρR^4 is constant and that $\dot{R}^2 + 1 = B^2/R^2$ for some constant B. Determine this constant.

(3.4) We take as boundary condition that $R = 0$ at $t = 0$. Show that R as a function of t is given by

$$R(t) = \sqrt{2Bt - t^2}\,. \tag{15}$$

Give the graph of the function $R(t)$. Let ρ_0 and R_0 be the values of the functions ρ and R at the present time $t = t_0$. Show that this universe has a finite lifetime and determine this lifetime in terms of ρ_0, R_0 and Newton's constant G.